

MODIFIED SCATTERING FOR THE KLEIN-GORDON EQUATION WITH THE CRITICAL NONLINEARITY IN THREE DIMENSIONS

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ABSTRACT. In this paper, we consider the final state problem for the nonlinear Klein-Gordon equation (NLKG) with a critical nonlinearity in three space dimensions: $(\square + 1)u = \lambda|u|^{2/3}u$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, where $\square = \partial_t^2 - \Delta$ is d'Alembertian. We prove that for a given asymptotic profile u_{ap} , there exists a solution u to (NLKG) which converges to u_{ap} as $t \rightarrow \infty$. Here the asymptotic profile u_{ap} is given by the leading term of the solution to the linear Klein-Gordon equation with a logarithmic phase correction. Construction of a suitable approximate solution is based on the combination of Fourier series expansion for the nonlinearity used in our previous paper [22] and smooth modification of phase correction by Ginibre-Ozawa [6].

1. INTRODUCTION

This paper is devoted to the study of the final state problem for the nonlinear Klein-Gordon equation with a critical nonlinearity in three space dimensions:

$$(1.1) \quad \begin{cases} (\square + 1)u = \lambda|u|^{2/3}u & t \in \mathbb{R}, x \in \mathbb{R}^3, \\ u - u_{\text{ap}} \rightarrow 0 & \text{in } L^2 \text{ as } t \rightarrow +\infty, \end{cases}$$

where $\square = \partial_t^2 - \Delta$ is d'Alembertian, $u : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is an unknown function, $u_{\text{ap}} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a given function, and λ is a non-zero real constant. The aim of this paper is to find a proper choice of the function u_{ap} so that the equation (1.1) admits a nontrivial solution. In other words, we want to determine a “right” asymptotic behavior which actually takes place. This is a continuation of our previous study of the two dimensional case in [23].

Let us briefly review the known results on the global existence and long time behavior of solution to the more general nonlinear Klein-Gordon equation

$$(1.2) \quad (\square + 1)u = \lambda|u|^{p-1}u, \quad t \in \mathbb{R}, x \in \mathbb{R}^d,$$

where $p > 1$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Since the point-wise decay of solution to the linear Klein-Gordon equation is $O(t^{-d/2})$ as $t \rightarrow \infty$, the linear scattering theory indicates that the power $p = 1 + 2/d$ will be a borderline between the short and long range scattering theories. This formal observation was firstly justified by Glassey [7], Matsumura [24] and Georgiev and Yordanov [5] for $p \leq 1 + 2/d$. More precisely, they proved that solutions to (1.2) do not scatter to the solution to the linear Klein-Gordon equation if $1 < p \leq 1 + 2/d$.

2000 *Mathematics Subject Classification.* Primary 35L71; Secondary 35B40, 81Q05.

Key words and phrases. scattering problem.

Later, Georgiev and Lecente [4] obtained a point-wise decay estimates for small solutions to the (1.2) for $p > 1 + 2/d$ with $d = 1, 2, 3$ by using the vector field approach by Klainerman [16]. Furthermore, Hayashi and Naumkin [10] proved that small solutions to (1.2) scatter to the solution to the linear Klein-Gordon equation if $p > 1 + 2/d$ and $d = 1, 2$. Notice that it is an still open problem for the asymptotic behavior of small solution to (1.2) when p is close to $1 + 2/d$ and $n \geq 3$. See [9, 16, 28, 29, 30, 32] for the small data scattering when $n \geq 3$ and p is large.

For the critical case $p = 1 + 2/d$ and $d = 1$, Georgiev and Yordanov [5] studied the point-wise decay of a solution to the initial value problem. Delort [1] obtained an asymptotic profile of a global solution to the equation. His proof is based on hyperbolic coordinates and the compactness of the support of the initial data was assumed. See also Lindblad and Soffer [18] for the alternative proof of [1]. The compact support assumption in [1] was later removed by Hayashi and Naumkin in [8] by using the vector field approach.

Recently, the authors [23] consider (1.2) with $p = 1 + 2/d$ and $d = 2$ and specify an asymptotic profile u_{ap} that allows a unique solution u which converges to u_{ap} as $t \rightarrow \infty$. The asymptotic profile u_{ap} has the same form as in the $d = 1$ case. Namely, it is the leading term of the solution to the linear Klein-Gordon equation with a logarithmic phase correction. The key ingredient is to extract a resonance term by means of Fourier series expansion of the nonlinearity. In this paper, we consider (1.1), that is, a similar final value problem for (1.2) with $p = 1 + 2/d$ and $d = 3$. Because the power becomes a fractional number, the argument in the two dimensional case [23] is not directly applicable. To deal with the nonlinearity, we use the argument in Ginibre-Ozawa [6].

Let us introduce the asymptotic profile u_{ap} which we work with. To this end, we first recall that the leading term of a solution to the linear Klein-Gordon equation

$$\begin{cases} (\square + 1)v = 0 & t \in \mathbb{R}, x \in \mathbb{R}^3, \\ v(0, x) = \phi_0(x), \quad \partial_t v(0, x) = \phi_1(x) & x \in \mathbb{R}^3 \end{cases}$$

is given by

$$t^{-\frac{3}{2}} \mathbf{1}_{\{|x| < t\}}(t, x) \langle \mu \rangle^{\frac{3}{2}} \rho(\mu) \operatorname{Re} e^{i(\langle \mu \rangle^{-1} t + \beta(\mu))},$$

where $\mu = \mu(t, x) := x / \sqrt{t^2 - |x|^2}$, $\mathbf{1}_{\Omega}(t, x)$ is the characteristic function supported on $\Omega \subset \mathbb{R}^{1+3}$, and $\rho \geq 0$ and $\beta \in [0, 2\pi)$ are given by the relation

$$\rho(\mu) e^{i\beta(\mu)} = e^{-i\frac{\pi}{4}} (\langle \mu \rangle \hat{\phi}_0(\mu) - i \hat{\phi}_1(\mu)),$$

see [11] for instance.

For given final state (ϕ_0, ϕ_1) , we define the asymptotic profile u_{ap} by

$$(1.3) \quad u_{\text{ap}}(t, x) := t^{-\frac{3}{2}} \mathbf{1}_{\{|x| < t\}}(t, x) \langle \mu \rangle^{\frac{3}{2}} \rho(\mu) \operatorname{Re} e^{i(\langle \mu \rangle^{-1} t + \Psi(\mu) \log t + \beta(\mu))},$$

where the phase correction term is given by

$$(1.4) \quad \Psi(\mu) = -\frac{\lambda \Gamma(\frac{11}{6})}{\sqrt{\pi} \Gamma(\frac{7}{3})} \rho(\mu)^{\frac{2}{3}}.$$

Remark that the coefficient comes from the first Fourier-cosine coefficient of a 2π -periodic function $|\cos \theta|^{2/3} \cos \theta$. The final state (ϕ_0, ϕ_1) is taken from

the function space Y defined by

$$\begin{aligned} Y &:= \{(\phi_0, \phi_1) \in \mathcal{S}'(\mathbb{R}^3) \times \mathcal{S}'(\mathbb{R}^3); \|(\phi_0, \phi_1)\|_Y < \infty\}, \\ \|(\phi_0, \phi_1)\|_Y &:= \|\phi_0\|_{H_x^2} + \|x\phi_0\|_{H_x^3} + \|x^2\phi_0\|_{H_x^4} \\ &\quad + \|\phi_1\|_{H_x^1} + \|x\phi_1\|_{H_x^2} + \|x^2\phi_1\|_{H_x^3}. \end{aligned}$$

The main result in this paper is as follows.

Theorem 1.1. *Let $(\phi_0, \phi_1) \in Y$. For $3/4 < \gamma < 5/6$, there exists $\varepsilon = \varepsilon(\gamma) > 0$ such that if $\|\langle \cdot \rangle^{3/2} \rho\|_{L_{\mu}^\infty} < \varepsilon$ then there exist $T \geq 3$ and a unique solution $u(t)$ for the equation (1.1) satisfying*

$$(1.5) \quad u \in C([T, \infty); H_x^{\frac{1}{2}}),$$

$$\sup_{t \geq T} t^\gamma \left(\|u - u_{\text{ap}}\|_{L^\infty((t, \infty); H_x^{\frac{1}{2}})} + \|u - u_{\text{ap}}\|_{L^{\frac{10}{3}}((t, \infty); L_x^{\frac{10}{3}})} \right) < \infty,$$

where the asymptotic profile u_{ap} is defined from (ϕ_0, ϕ_1) via (1.3).

Remark 1.2. The same result holds true for equations with a general critical nonlinearity $F(u) : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $F(\lambda u) = \lambda^{5/3} F(u)$ for all $\lambda > 0$ and $u \in \mathbb{R}$. See Remark 2.5 below for the detail.

Remark 1.3. Concerning the scattering results for the Klein-Gordon equation with the critical quasilinear nonlinearity, the readers can consult Moriyama [25], Katayama [12], Sunagawa [33] for one dimensional cubic case and Ozawa, Tsutaya and Tsutsumi [27], Delort, Fang and Xue [2], Kawahara and Sunagawa [14], Katayama, Ozawa and Sunagawa [13] for the two dimensional quadratic case.

The rest of the paper is organized as follows. In Section 2, we exhibit the outline of the proof of Theorem 1.1. We construct a solution with the desired property by applying the contraction principle to the integral equation of Yang-Feldman type associated with (1.1) around a suitable approximate solution. The crucial points of the proof are summarized as Propositions 2.1 and 2.4. Then, we prove Proposition 2.1 in Section 3 and Proposition 2.4 in Section 4

2. OUTLINE OF THE PROOF OF THEOREM 1.1

In this section, we give an outline of the proof of Theorem 1.1.

2.1. On the solvability of the final state problem. We first remark that solvability of (1.1) is reduced to the appropriateness of the choice of the asymptotic behavior u_{ap} .

Let $A(t, x)$ be a given asymptotic profile of a solution to (1.1). We show that if $A(t, x)$ is well-chosen then we obtain a solution which asymptotically behaves like $A(t, x)$. Let $N(u) = \lambda|u|^{2/3}u$.

Proposition 2.1. *For any $\gamma > 3/4$, there exists $\eta = \eta(\gamma) > 0$ such that if $A(t, x)$ satisfies*

$$(2.1) \quad \sup_{t \geq T_0} t^{\frac{3}{2}} \|A(t)\|_{L_x^\infty} \leq \eta,$$

$$(2.2) \quad \sup_{t \geq T_0} t^{1+\gamma} \|(\square + 1)A(t) - N(A(t))\|_{L_x^2} < \infty$$

for some $T_0 \geq 3$, then there exist $T \geq T_0$ and a unique solution $u \in C([T, \infty); L_x^2)$ for the equation (1.1) satisfying

$$(2.3) \quad \sup_{t \geq T} t^\gamma \left(\|u - A\|_{L^\infty((t, \infty); H_x^{\frac{1}{2}})} + \|u - A\|_{L^{\frac{10}{3}}((t, \infty); L_x^{\frac{10}{3}})} \right) < \infty$$

for the same γ .

The proposition will be proven in Section 3.

2.2. Choice of an appropriate asymptotic profile. An easy choice is $A = u_{\text{ap}}$. However, it does not work well. Hence, we choose a suitable A that satisfies the assumptions (2.1) and (2.2) and

$$(2.4) \quad \sup_{t \geq T} t^\gamma \left(\|u_{\text{ap}} - A\|_{L^\infty((t, \infty); H_x^{\frac{1}{2}})} + \|u_{\text{ap}} - A\|_{L^{\frac{10}{3}}((t, \infty); L_x^{\frac{10}{3}})} \right) < \infty.$$

Then, the solution obtained by means of Proposition 2.1 from A possesses the desired asymptotics.

The obstacle in three-dimensional case lies in the fact that the phase correction term Ψ , given in (1.4), has the fractional power term $\rho^{2/3}$. The power comes from the nonlinearity. Notice that because of the fractional power, we may not estimate $(\square + 1)u_{\text{ap}}$, in general. To overcome the difficulty, we exploit the argument in Ginibre-Ozawa [6]. We introduce a modified phase corrector

$$\tilde{\Psi}(s, \mu) := -\frac{\lambda \Gamma(\frac{11}{6})}{\sqrt{\pi} \Gamma(\frac{7}{3})} \tilde{\rho}(s, \mu)^{\frac{2}{3}}, \quad \tilde{\rho}(s, \mu) = \sqrt{\rho(\mu)^2 + s^{-1} \langle \mu \rangle^{-3}}.$$

and an auxiliary approximate solution

$$(2.5) \quad \tilde{u}_{\text{ap}}(t, x) = t^{-\frac{3}{2}} \mathbf{1}_{\{|x| < t\}}(t, x) \langle \mu \rangle^{\frac{3}{2}} \rho(\mu) \operatorname{Re} e^{i(\alpha(t, \mu) + \beta(\mu))},$$

where $\mu = \mu(t, x) := x / \sqrt{t^2 - |x|^2}$ and

$$\alpha(t, \mu) := \langle \mu \rangle^{-1} t + \tilde{\Psi}(t, \mu) \log t.$$

We will see the error from the modification is acceptable.

Starting from the modified asymptotic profile \tilde{u}_{ap} , we construct the profile A as in the two dimensional case [23]. This is the idea of the proof. For readers' convenience, we recall the construction of A . Since $N(\tilde{u}_{\text{ap}}(t))$ is $O(t^{-1})$ in L_x^2 as $t \rightarrow \infty$, we have to, at least, find the main parts of it and cancel them out, otherwise (2.2) fails. In [23], a Fourier series expansion is introduced for this purpose. Here, we split

$$\begin{aligned} N(\tilde{u}_{\text{ap}}) &= \lambda t^{-\frac{5}{2}} \mathbf{1}_{\{|x| < t\}} \langle \mu \rangle^{\frac{5}{2}} \rho(\mu)^{\frac{5}{3}} |\cos(\alpha + \beta)|^{\frac{2}{3}} \cos(\alpha + \beta) \\ &= \lambda c_1 t^{-\frac{5}{2}} \mathbf{1}_{\{|x| < t\}} \langle \mu \rangle^{\frac{5}{2}} \rho(\mu)^{\frac{5}{3}} \operatorname{Re} e^{i(\alpha + \beta)} \\ &\quad + \lambda t^{-\frac{5}{2}} \mathbf{1}_{\{|x| < t\}} \langle \mu \rangle^{\frac{5}{2}} \rho(\mu)^{\frac{5}{3}} \sum_{n=3}^{\infty} c_n \operatorname{Re} e^{in(\alpha + \beta)} \\ &=: N_r + N_{\text{nr}}, \end{aligned} \tag{2.6}$$

where c_n is the Fourier-cosine coefficient of $|\cos \theta|^{\frac{2}{3}} \cos \theta$. We employ the following estimate on the coefficient.

Lemma 2.2 ([21]). *Let $c_n := \frac{1}{\pi} \int_{-\pi}^{\pi} |\cos \theta|^{\frac{2}{3}} \cos \theta \cos n\theta d\theta$ for $n \geq 0$. Then, $c_n = 0$ for even n and*

$$c_n = \frac{2(-1)^{\frac{n-1}{2}} \Gamma(\frac{11}{6}) \Gamma(\frac{3n-5}{6})}{\sqrt{\pi} \Gamma(-\frac{1}{3}) \Gamma(\frac{3n+11}{6})}$$

for odd n . In particular, $c_n = O(n^{-8/3})$ as $n \rightarrow \infty$.

Thanks to the choice of the phase function $\tilde{\Psi}$, the resonance part N_r is close to $(\square + 1)\tilde{u}_{\text{ap}}$ (See Proposition 4.5). To cancel out N_{nr} , we introduce

$$(2.7) \quad \tilde{v}_{\text{ap}}(t, x) := \sum_{n=2}^{\infty} v_n(t, \mu(t, x)),$$

with

$$(2.8) \quad v_n(s, \mu) := -\frac{\lambda c_n}{n^2 - 1} t^{-\frac{5}{2}} \langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) \operatorname{Re}(e^{in(\alpha(s, \mu) + \beta(\mu))}).$$

It will turn out that the non-resonance part N_{nr} is successfully canceled out by \tilde{v}_{ap} (See Proposition 4.6).

Remark 2.3. This kind of approximation was introduced in Hörmander [11] for the Klein-Gordon equation with *polynomial* nonlinearity in (u, \bar{u}) . See also [26, 31] for the nonlinear Schrödinger equation with polynomial nonlinearity in (u, \bar{u}) .

Based on the above observation, we will show the following proposition.

Proposition 2.4. *Let $A = \tilde{u}_{\text{ap}} + \tilde{v}_{\text{ap}}$, where \tilde{u}_{ap} and \tilde{v}_{ap} are given in (2.5) and (2.7), respectively. Then, (2.2) and (2.4) holds for any $\gamma < 5/6$ and $T_0 \geq 3$. Furthermore, for any $\eta > 0$ and $\gamma < 5/6$, there exists ε such that if $\|\langle \cdot \rangle^{3/2} \rho\|_{L_{\mu}^{\infty}} \leq \varepsilon$ then A satisfies (2.1) for some $T_0 \geq 3$.*

Together with Proposition 2.1, this proposition implies Theorem 1.1. Section 4 is devoted to the proof of the above proposition.

Remark 2.5. Let us consider a generalization of Theorem 1.1 to any real-valued nonlinearity satisfying $F(\lambda u) = \lambda^{5/3} F(u)$ for any $\lambda > 0$ and $u \in \mathbb{R}$. Notice that this class of nonlinearity is written as $F(u) = \lambda_1 |u|^{\frac{2}{3}} u + \lambda_2 |u|^{\frac{5}{3}}$. Theorem 1.1 corresponds to the case where $\lambda_2 = 0$. By means of the following lemma, we see that the nonlinearity $\lambda_2 |u|^{5/3}$ does not contain a resonant part, and so that we can treat the above general nonlinearity by the same argument.

Lemma 2.6. *Let $\tilde{c}_n := \frac{1}{\pi} \int_{-\pi}^{\pi} |\cos \theta|^{\frac{5}{3}} \cos n\theta d\theta$ for $n \geq 0$. Then, $\tilde{c}_n = 0$ for odd n and*

$$\tilde{c}_n = \frac{2(-1)^{\frac{n}{2}} \Gamma(\frac{4}{3}) \Gamma(\frac{3n-5}{6})}{\sqrt{\pi} \Gamma(-\frac{5}{6}) \Gamma(\frac{3n+11}{6})}$$

for even n . In particular, $\tilde{c}_n = O(n^{-8/3})$ as $n \rightarrow \infty$.

Proof. The proof is similar to Lemma 2.2. See, [21]. □

3. PROOF OF PROPOSITION 2.1

In this section, we prove Proposition 2.1. The proof is essentially the same as in the two dimensional case [23]. The following inhomogeneous Strichartz estimates associated with the Klein-Gordon equation is crucial for the proof. Let

$$(3.1) \quad \mathcal{G}[g](t) := \int_t^\infty \sin((t-\tau)\sqrt{1-\Delta})(1-\Delta)^{-1/2}g(\tau)d\tau.$$

Lemma 3.1. *Let $2 \leq q \leq 6$ and $2/p + 3/q = 3/2$. Then we have*

$$\begin{aligned} \|\mathcal{G}[g]\|_{L_t^p([T,\infty),L_x^q)} &\leq C\|(1-\Delta)^{\frac{1}{2}(\frac{3}{2}-\frac{5}{q})}g\|_{L_t^{p'}([T,\infty),L_x^{q'})}, \\ \|\mathcal{G}[g]\|_{L_t^\infty([T,\infty),L_x^2)} &\leq C\|(1-\Delta)^{\frac{1}{2}(\frac{1}{4}-\frac{5}{2q})}g\|_{L_t^{p'}([T,\infty),L_x^{q'})}, \\ \|\mathcal{G}[g]\|_{L_t^p([T,\infty),L_x^q)} &\leq C\|(1-\Delta)^{\frac{1}{2}(\frac{1}{4}-\frac{5}{2q})}g\|_{L_t^1([T,\infty),L_x^2)}. \end{aligned}$$

Proof. The above inequalities follow from combination of the L^p - L^q estimate for the solution to the Klein-Gordon equation by [19] with the duality argument by [35] for the non-endpoint case $q \neq 6$ and the argument by [15] for the endpoint case $q = 6$. Since the proof is now standard, we omit the detail. \square

Proof of Proposition 2.1. We introduce

$$X_T = \{w \in C([T, \infty); L_x^2); \|w\|_{X_T} < \infty\}$$

for $T \geq 3$, where

$$\|w\|_{X_T} = \sup_{t \geq T} t^\gamma \left(\|w\|_{L^\infty((t,\infty);H_x^{\frac{1}{2}})} + \|w\|_{L^{\frac{10}{3}}((t,\infty);L_x^{\frac{10}{3}})} \right).$$

For $R > 0$ and $T > 0$, we define

$$\tilde{X}_T(R) = \{w \in C([T, \infty); L_x^2); \|w\|_{X_T} \leq R\}.$$

The function space X_T is a Banach space with the norm $\|\cdot\|_{X_T}$ and $\tilde{X}_T(\rho)$ is a complete metric space with the $\|\cdot\|_{X_T}$ -metric.

We put $v = u - A$. Then the equation (1.1) is equivalent to

$$(3.2) \quad (\square + 1)v = N(v + A) - N(A) - F,$$

where

$$F := (\square + 1)A - N(A)$$

The associate integral equation to the equation (3.2) is

$$(3.3) \quad v = \mathcal{G}[\{N(v + A) - N(A)\} - F],$$

where \mathcal{G} is given by (3.1). It suffices to show the existence of a unique solution v to the equation (3.3) in X_T for suitable $\eta > 0$ and $T \geq T_0$. We prove this assertion by the contraction argument. Define the nonlinear operator Φ by

$$\Phi v := \mathcal{G}[\{N(v + A) - N(A)\} - F]$$

for $v \in \tilde{X}_T(R)$. We show that Φ is a contraction map on $\tilde{X}_T(\rho)$ if $R > 0$, $T \geq T_0$, and $\eta > 0$ are suitably chosen. Let $v \in \tilde{X}_T(R)$ and $t \geq T$. By the assumptions and Lemma 3.1, we see

$$\begin{aligned}
& \|(\Phi v)(t)\|_{L^\infty((t,\infty);H_x^{\frac{1}{2}})} + \|\Phi v\|_{L^{\frac{10}{3}}((t,\infty);L_x^{\frac{10}{3}})} \\
& \leq C(\|v\|^{\frac{2}{3}}v\|_{L^{\frac{10}{7}}((t,\infty);L_x^{\frac{10}{7}})} \\
& \quad + \|(1-\Delta)^{-\frac{1}{4}}\{|v+A|^{\frac{2}{3}}(v+A) - |v|^{\frac{2}{3}}v - |A|^{\frac{2}{3}}A\}\|_{L^1((t,\infty);L_x^2)} \\
& \quad + \|(1-\Delta)^{-\frac{1}{4}}F\|_{L^1((t,\infty);L_x^2)}) \\
& \leq C\left\{\|v\|^{\frac{2}{3}}_{L^{\frac{10}{3}}((t,\infty);L_x^{\frac{10}{3}})}\left(\int_t^\infty \|v(\tau)\|_{L_x^2}^2 d\tau\right)^{\frac{1}{2}} + \int_t^\infty \|A(\tau)\|_{L_x^\infty}^{\frac{2}{3}}\|v(\tau)\|_{L_x^2} d\tau\right. \\
& \quad \left.+ \int_t^\infty \|F(\tau)\|_{L_x^2} d\tau\right\} \\
& \leq C\left\{R^{\frac{2}{3}}t^{-\frac{2}{3}\gamma}\left(\int_t^\infty R^2\tau^{-2\gamma} d\tau\right)^{\frac{1}{2}} + \int_t^\infty \eta^{\frac{2}{3}}R\tau^{-1-\gamma} d\tau + \int_t^\infty M\tau^{-1-\gamma} d\tau\right\} \\
& \leq Ct^{-\gamma}(R^{\frac{5}{3}}t^{-\frac{2}{3}\gamma+\frac{1}{2}} + R\eta^{\frac{2}{3}} + M),
\end{aligned}$$

where M is an upper bound on the right hand side of (2.2). Therefore we obtain

$$(3.4) \quad \|\Phi v\|_{X_T} \leq C_1(R^{\frac{5}{3}}T^{-\frac{2}{3}\gamma+\frac{1}{2}} + R\eta^{\frac{2}{3}} + M).$$

In the same way as above, for $v_1, v_2 \in \tilde{X}_T(R)$, we can show

$$\begin{aligned}
(3.5) \quad & \|\Phi v_1 - \Phi v_2\|_{X_T} \\
& \leq C_2((\|v_1\|_{X_T}^{\frac{2}{3}} + \|v_2\|_{X_T}^{\frac{2}{3}})T^{-\frac{2}{3}\gamma+\frac{1}{2}} + \eta^{\frac{2}{3}})\|v_1 - v_2\|_{X_T} \\
& \leq C_2(R^{\frac{2}{3}}T^{-\frac{2}{3}\gamma+\frac{1}{2}} + \eta^{\frac{2}{3}})\|v_1 - v_2\|_{X_T}.
\end{aligned}$$

We first fix R so that $C_1M \leq R/2$. Then, using the fact that $\gamma > 3/4$, we are able to choose a sufficiently large $T > 0$ and a sufficiently small $\eta > 0$ such that

$$\begin{aligned}
C_1(R^{\frac{5}{3}}T^{-\frac{2}{3}\gamma+\frac{1}{2}} + R\eta^{\frac{2}{3}} + \eta) & \leq R, \\
C_2(R^{\frac{2}{3}}T^{-\frac{2}{3}\gamma+\frac{1}{2}} + \eta^{\frac{2}{3}}) & \leq \frac{1}{2},
\end{aligned}$$

For such R, T, η , there exists a unique solution to the integral equation (3.3) in $\tilde{X}_T(\rho)$. The uniqueness of solutions to the equation (3.3) in X_T follows from the first inequality of the estimate (3.5) for solutions $v_1 \in X_T$ and $v_2 \in X_T$. Hence the equation (3.3) has a unique solution in X_T . This completes the proof of Proposition 2.1. \square

4. PROOF OF PROPOSITION 2.4

In this section, we prove Proposition 2.4. Since (2.1) is trivial, we prove (2.4) and (2.2) in Sections 4.2 and 4.3, respectively, after preparing preliminary estimates in Section 4.1. Hereafter we always restrict our attention to the region $|x| < t$ and $t \geq 3$.

We introduce new variables (s, μ) by $s = t$ and $\mu = x/\sqrt{t^2 - |x|^2}$. Then, we have

$$(4.1) \quad \partial_t = \partial_s - s^{-1} \langle \mu \rangle^2 \mu \cdot \nabla_\mu,$$

$$(4.2) \quad \partial_{x_i} = s^{-1} \langle \mu \rangle \partial_{\mu_i} + s^{-1} \langle \mu \rangle \mu_i \mu \cdot \nabla_\mu$$

and

$$(4.3) \quad \begin{aligned} \square &= \partial_s^2 - 2s^{-1} \langle \mu \rangle^2 \mu \cdot \nabla_\mu \partial_s - s^{-2} \langle \mu \rangle^2 \Delta_\mu \\ &\quad - 3s^{-2} \langle \mu \rangle^2 \mu \cdot \nabla_\mu - s^{-2} \langle \mu \rangle^2 \sum_{1 \leq i, j \leq 3} \mu_i \mu_j \partial_i \partial_j. \end{aligned}$$

Also remark that

$$(4.4) \quad \|f(t, \mu(t, x))\|_{L_x^p(|x| < t)} = s^{\frac{3}{p}} \left\| \langle \mu \rangle^{-\frac{5}{p}} f(s, \mu) \right\|_{L_\mu^p(\mathbb{R}^3)}$$

for any $p \in (0, \infty]$.

4.1. Preliminaries. We collect preliminary estimates.

Lemma 4.1. *For $n, m \in \mathbb{R}$,*

$$\begin{aligned} (\square_{t,x} + 1)(s^{-m} e^{in\langle \mu \rangle^{-1}s}) &= -(n^2 - 1)s^{-m} e^{in\langle \mu \rangle^{-1}s} \\ &\quad - in(2m - d)s^{-m-1} \langle \mu \rangle e^{in\langle \mu \rangle^{-1}s} + m(m+1)s^{-m-2} e^{in\langle \mu \rangle^{-1}s}, \end{aligned}$$

where $d = 3$ is the spatial dimension.

Proof. It follows by direct calculation. \square

Recall that $\tilde{\rho}(s, \mu) = \sqrt{\rho(\mu)^2 + s^{-1} \langle \mu \rangle^{-3}}$. An elementary inequality

$$(4.5) \quad \max(\rho(\mu), s^{-\frac{1}{2}} \langle \mu \rangle^{-\frac{3}{2}}) \leq \tilde{\rho}(s, \mu) \leq \rho(\mu) + s^{-\frac{1}{2}} \langle \mu \rangle^{-\frac{3}{2}}$$

will be useful. The following will be used to estimate the error comes from the phase modification.

Lemma 4.2. *For $s \geq 3$, we have the following inequality:*

$$\rho(\mu)(\tilde{\rho}(s, \mu)^{\frac{2}{3}} - \rho(\mu)^{\frac{2}{3}}) \lesssim s^{-\frac{5}{6}} \langle \mu \rangle^{-\frac{5}{2}},$$

where the implicit constant is independent of ρ .

Proof. By a direct calculation, we obtain

$$\begin{aligned} &\rho(\mu)(\tilde{\rho}(s, \mu)^{\frac{2}{3}} - \rho(\mu)^{\frac{2}{3}}) \\ &= \frac{1}{3} \rho(\mu) \int_0^1 (\rho(\mu)^2 + \theta s^{-1} \langle \mu \rangle^{-3})^{-\frac{2}{3}} s^{-1} \langle \mu \rangle^{-3} d\theta \\ &\leq \frac{1}{3} \rho(\mu) \int_0^1 (\rho(\mu)^2)^{-\frac{1}{2}} (\theta s^{-1} \langle \mu \rangle^{-3})^{-\frac{1}{6}} s^{-1} \langle \mu \rangle^{-3} d\theta = C s^{-\frac{5}{6}} \langle \mu \rangle^{-\frac{5}{2}} \end{aligned}$$

for every $\mu \in \mathbb{R}^3$. Thus we obtain the desired inequality. \square

Now, we turn to the estimate of the derivatives of the modified phase part $\tilde{\Psi}(s, \rho) = -(\lambda c_1/2) \tilde{\rho}(s, \mu)^{2/3}$.

Lemma 4.3. *For $s \geq 3$, we have the following inequalities:*

$$(4.6) \quad |\partial_s \tilde{\Psi}(s, \mu)| \lesssim s^{-\frac{4}{3}} \langle \mu \rangle^{-1},$$

$$(4.7) \quad |\rho(\mu) \partial_s \tilde{\Psi}(s, \mu)| \lesssim s^{-1-\frac{5}{6}} \langle \mu \rangle^{-\frac{5}{2}},$$

$$(4.8) \quad |\rho(\mu) (\partial_s \tilde{\Psi}(s, \mu))^2| \lesssim s^{-2-\frac{7}{6}} \langle \mu \rangle^{-\frac{7}{2}},$$

$$(4.9) \quad |\rho(\mu)^{\frac{2}{3}} \tilde{\Psi}(s, \mu) \partial_s \tilde{\Psi}(s, \mu)| \lesssim s^{-2} \langle \mu \rangle^{-3},$$

$$(4.10) \quad |\rho(\mu) \partial_s^2 \tilde{\Psi}(s, \mu)| \lesssim s^{-2-\frac{5}{6}} \langle \mu \rangle^{-\frac{5}{2}},$$

where the implicit constants are independent of ρ .

Proof. The first four inequalities (4.6)-(4.9) follow from

$$\partial_s \tilde{\Psi}(s, \mu) = C s^{-2} \langle \mu \rangle^{-3} \tilde{\rho}(s, \mu)^{-\frac{4}{3}},$$

and (4.5). Similarly,

$$\partial_s^2 \tilde{\Psi}(s, \mu) = C_1 s^{-3} \langle \mu \rangle^{-3} \tilde{\rho}(s, \mu)^{-\frac{4}{3}} + C_2 s^{-4} \langle \mu \rangle^{-6} \tilde{\rho}(s, \mu)^{-\frac{10}{3}},$$

yields the last inequality (4.10). \square

Lemma 4.4. *For $s \geq 3$, we have the following inequalities:*

$$(4.11) \quad |\partial_{\mu_j} \tilde{\Psi}(s, \mu)| \lesssim s^{\frac{1}{6}} \langle \mu \rangle^{\frac{1}{2}} |\partial_{\mu_j} \rho(\mu)| + s^{-\frac{1}{3}} \langle \mu \rangle^{-2},$$

$$(4.12) \quad |\rho(\mu)^{\frac{2}{3}} \partial_{\mu_j} \tilde{\Psi}(s, \mu) \partial_{\mu_k} \tilde{\Psi}(s, \mu)| \lesssim |\nabla_\mu \rho(\mu)|^2 + s^{-1} \langle \mu \rangle^{-5},$$

$$(4.13) \quad |\rho(\mu) \partial_{\mu_j} \partial_{\mu_k} \tilde{\Psi}(s, \mu)| \lesssim \rho(\mu)^{\frac{2}{3}} |\nabla_\mu^2 \rho(\mu)| + s^{\frac{1}{6}} \langle \mu \rangle^{\frac{1}{2}} |\nabla_\mu \rho(\mu)|^2 + s^{-\frac{1}{3}} \langle \mu \rangle^{-3} \rho(\mu),$$

$$(4.14) \quad |\rho(\mu) \partial_{\mu_j} \partial_s \tilde{\Psi}(s, \mu)| \lesssim s^{-\frac{4}{3}} \langle \mu \rangle^{-1} |\partial_{\mu_j} \rho(\mu)| + s^{-\frac{4}{3}} \langle \mu \rangle^{-2} \rho(\mu),$$

where the implicit constants are independent of ρ .

Proof. The first two inequalities (4.11) and (4.12) follow from

$$\partial_{\mu_j} \tilde{\Psi} = C_1 \tilde{\rho}(s, \mu)^{-\frac{4}{3}} (\rho(\mu) \partial_{\mu_j} \rho(\mu) - 3s^{-1} \langle \mu \rangle^{-5} \mu_j)$$

and (4.5). To obtain the inequality (4.13), we use

$$\begin{aligned} \partial_{\mu_j} \partial_{\mu_k} \tilde{\Psi} &= C_2 \tilde{\rho}(s, \mu)^{-\frac{10}{3}} (\rho(\mu) \partial_{\mu_j} \rho(\mu) - 3s^{-1} \langle \mu \rangle^{-5} \mu_j) \\ &\quad \times (\rho(\mu) \partial_{\mu_k} \rho(\mu) - 3s^{-1} \langle \mu \rangle^{-5} \mu_k) \\ &\quad + C_1 \tilde{\rho}(s, \mu)^{-\frac{4}{3}} (\partial_{\mu_j} \rho(\mu) \partial_{\mu_k} \rho(\mu) + \rho(\mu) \partial_{\mu_j} \partial_{\mu_k} \rho(\mu)) \\ &\quad + C_1 \tilde{\rho}(s, \mu)^{-\frac{4}{3}} (15s^{-1} \langle \mu \rangle^{-7} \mu_j \mu_k - 3s^{-1} \langle \mu \rangle^{-5} \delta_{jk}), \end{aligned}$$

where δ_{jk} is the Kronecker delta. The inequality (4.14) is a consequence of

$$\begin{aligned} \partial_{\mu_j} \partial_s \tilde{\Psi} &= C_3 s^{-2} \langle \mu \rangle^{-3} \tilde{\rho}(s, \mu)^{-\frac{10}{3}} (\rho(\mu) \partial_{\mu_j} \rho(\mu) - 3s^{-1} \langle \mu \rangle^{-5} \mu_j) \\ &\quad + 3C_1 s^{-2} \tilde{\rho}(s, \mu)^{-\frac{4}{3}} \langle \mu \rangle^{-5} \mu_j. \end{aligned}$$

This completes the proof of Lemma 4.4. \square

4.2. Proof of (2.4). We now show that A satisfies (2.4) for $\gamma < 5/6$.

Proof. Since

$$|e^{-\frac{ic_1\lambda}{2}\tilde{\rho}(s,\mu)^{\frac{2}{3}}\log t} - e^{-\frac{ic_1\lambda}{2}\rho(\mu)^{\frac{2}{3}}\log t}| \lesssim C(\tilde{\rho}(s,\mu)^{\frac{2}{3}} - \rho(\mu)^{\frac{2}{3}})\log t$$

for $t \geq 3$, we deduce from (4.4) and Lemma 4.2 that

$$\|\tilde{u}_{\text{ap}} - u_{\text{ap}}\|_{L_x^2} \lesssim (\log t) \left\| \langle \mu \rangle^{-1} \rho(\mu) (\tilde{\rho}(s,\mu)^{\frac{2}{3}} - \rho(\mu)^{\frac{2}{3}}) \right\|_{L_\mu^2(\mathbb{R}^3)} \lesssim t^{-\frac{5}{6}} \log t$$

and

$$\|\tilde{u}_{\text{ap}} - u_{\text{ap}}\|_{L_x^{\frac{10}{3}}} \lesssim t^{-\frac{3}{5} - \frac{5}{6}} \log t.$$

Further, we see from (4.5) that

$$\|\tilde{v}_{\text{ap}}\|_{L_x^2(|x|<t)} \lesssim t^{-1} \|\rho\|_{L_\mu^{\frac{10}{3}}}^{\frac{5}{3}} + t^{-\frac{4}{3}} \|\rho\|_{L_\mu^2}$$

and

$$\|\tilde{v}_{\text{ap}}\|_{L_x^{\frac{10}{3}}(|x|<t)} \lesssim t^{-\frac{3}{5}} (t^{-1} \left\| \langle \mu \rangle^{\frac{3}{5}} \rho \right\|_{L_\mu^{\frac{50}{9}}}^{\frac{5}{3}} + t^{-\frac{4}{3}} \|\rho\|_{L_\mu^2}).$$

Similarly, we have

$$\|\nabla_x \tilde{u}_{\text{ap}} - \nabla_x u_{\text{ap}}\|_{L_x^2} + \|\nabla_x \tilde{v}_{\text{ap}}\|_{L_x^2} \lesssim t^{-\frac{5}{6}} (\log t) \langle \|(\phi_0, \phi_1)\|_Y \rangle^{\frac{7}{3}}.$$

Indeed, in view of (4.2), the leading term with respect to t appears only when the derivative ∇_x hits $e^{in\langle\mu\rangle^{-1}t}$. Furthermore, in that case, $\nabla_x e^{in\langle\mu\rangle^{-1}t} = -in\mu e^{i\langle\mu\rangle^{-1}t}$ and so the estimate is essentially the same.

Combining these estimates, we conclude that A satisfies (2.4) as long as $\gamma < 5/6$. \square

4.3. Proof of (2.2). To complete the proof of Proposition 2.4, we prove A satisfies the condition (2.2) for $\gamma < 5/6$. Note that

$$(\square + 1)A - N(A) = ((\square + 1)\tilde{u}_{\text{ap}} - N_{\text{r}}) + ((\square + 1)\tilde{v}_{\text{ap}} - N_{\text{nr}}) + (N(\tilde{u}_{\text{ap}}) - N(A)).$$

The third term of the right hand side is estimated as

$$\begin{aligned} \|N(A) - N(\tilde{u}_{\text{ap}})\|_{L_x^2} &\lesssim (\|\tilde{u}_{\text{ap}}\|_{L_x^\infty} + \|\tilde{v}_{\text{ap}}\|_{L_x^\infty})^{\frac{2}{3}} \|\tilde{u}_{\text{ap}}\|_{L_x^2} \\ &\lesssim t^{-2} \left\langle \left\| \langle \cdot \rangle^3 \rho e^{i\beta} \right\|_{H^2} \right\rangle^3. \end{aligned}$$

Hence, we estimate the first and the second terms, in what follows.

Proposition 4.5. *For $t \geq 3$,*

$$(4.15) \quad \|(\square_{t,x} + 1)\tilde{u}_{\text{ap}} - N_{\text{r}}\|_{L_x^2(|x|<t)} \lesssim t^{-1-\frac{5}{6}} (\log t) \langle \|(\phi_0, \phi_1)\|_Y \rangle^{\frac{7}{3}}$$

holds.

Proof. To show the inequality (4.15), we begin with the computation of the linear part:

$$(\square + 1)\tilde{u}_{\text{ap}} = \text{Re} \left[(\square + 1)(s^{-\frac{3}{2}} \langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta} e^{i\langle\mu\rangle^{-1}s + i\tilde{\Psi}(s,\mu)\log s}) \right].$$

We split

$$\begin{aligned}
& (\square + 1)(s^{-\frac{3}{2}}\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\langle\mu\rangle^{-1}s+i\tilde{\Psi}(s,\mu)\log s}) \\
& = ((\square + 1)s^{-\frac{3}{2}}e^{i\langle\mu\rangle^{-1}s})\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s} \\
& \quad + s^{-\frac{3}{2}}e^{i\langle\mu\rangle^{-1}s}\square(\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s}) \\
& \quad + 2\partial_t(s^{-\frac{3}{2}}e^{i\langle\mu\rangle^{-1}s})\partial_t(\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s}) \\
& \quad - 2\nabla_x(s^{-\frac{3}{2}}e^{i\langle\mu\rangle^{-1}s}) \cdot \nabla_x(\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s}) \\
& =: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.16}$$

In light of Lemma 4.1, we have

$$I_1 = \frac{15}{4}s^{-\frac{7}{2}}\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i(\alpha+\beta)}. \tag{4.17}$$

By (4.3), one sees that

$$\begin{aligned}
I_2 & = s^{-\frac{3}{2}}e^{i\langle\mu\rangle^{-1}s}\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}\partial_s^2e^{i\tilde{\Psi}(s,\mu)\log s} \\
& \quad - 2s^{-\frac{5}{2}}e^{i\langle\mu\rangle^{-1}s}\langle\mu\rangle^2\mu \cdot \nabla_\mu\partial_s(\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s}) \\
& \quad - s^{-\frac{7}{2}}e^{i\langle\mu\rangle^{-1}s}\langle\mu\rangle^2\Delta_\mu(\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s}) \\
& \quad - 3s^{-\frac{7}{2}}e^{i\langle\mu\rangle^{-1}s}\langle\mu\rangle^2\mu \cdot \nabla_\mu(\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s}) \\
& \quad - s^{-\frac{7}{2}}e^{i\langle\mu\rangle^{-1}s}\langle\mu\rangle^2 \sum_{1 \leq j,k \leq 3} \mu_j\mu_k\partial_j\partial_k(\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s}) \\
& =: J_1 + J_2 + J_3 + J_0 + J_4.
\end{aligned} \tag{4.18}$$

Moreover, since

$$\begin{aligned}
\partial_t(s^{-\frac{3}{2}}e^{i\langle\mu\rangle^{-1}s}) & = i\langle\mu\rangle s^{-\frac{3}{2}}e^{i\langle\mu\rangle^{-1}s} - \frac{3}{2}s^{-\frac{5}{2}}e^{i\langle\mu\rangle^{-1}s}, \\
\nabla_x e^{i\langle\mu\rangle^{-1}s} & = -i\mu e^{i\langle\mu\rangle^{-1}s},
\end{aligned}$$

we have

$$I_4 = 2is^{-\frac{5}{2}}e^{i\langle\mu\rangle^{-1}s}\langle\mu\rangle^3(\mu \cdot \nabla_\mu\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s}) \tag{4.19}$$

and

$$\begin{aligned}
I_3 & = -2s^{-\frac{5}{2}}\tilde{\Psi}(s,\mu)\langle\mu\rangle^{\frac{5}{2}}\rho(\mu)e^{i(\alpha+\beta)} \\
& \quad - 2s^{-\frac{3}{2}}(\log s)\partial_s\tilde{\Psi}(s,\mu)\langle\mu\rangle^{\frac{5}{2}}\rho(\mu)e^{i(\alpha+\beta)} \\
& \quad - 2is^{-\frac{5}{2}}e^{i\langle\mu\rangle^{-1}s}\langle\mu\rangle^3(\mu \cdot \nabla_\mu(\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s})) \\
& \quad - 3is^{-\frac{7}{2}}\tilde{\Psi}(s,\mu)\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i(\alpha+\beta)} \\
& \quad - 3is^{-\frac{5}{2}}(\log s)\partial_s\tilde{\Psi}(s,\mu)\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i(\alpha+\beta)} \\
& \quad + 3s^{-\frac{7}{2}}e^{i\langle\mu\rangle^{-1}s}\langle\mu\rangle^2(\mu \cdot \nabla_\mu(\langle\mu\rangle^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s,\mu)\log s})) \\
& =: -2s^{-\frac{5}{2}}\tilde{\Psi}(s,\mu)\langle\mu\rangle^{\frac{5}{2}}\rho(\mu)e^{i(\alpha+\beta)} + J_5 - I_4 + J_6 + J_7 - J_0.
\end{aligned} \tag{4.20}$$

From (4.16), (4.17), (4.18), (4.19) and (4.20) we reach to

$$\begin{aligned}
& (\square + 1)u_{\text{ap}} - N_r \\
& = \lambda c_1 s^{-\frac{5}{2}} \langle \mu \rangle^{\frac{5}{2}} (\tilde{\rho}(s, \mu)^{\frac{2}{3}} - \rho(\mu)^{\frac{2}{3}}) \rho(\mu) \operatorname{Re}(e^{i(\alpha+\beta)}) \\
(4.21) \quad & + \operatorname{Re} I_1 + \sum_{k=1}^7 \operatorname{Re} J_k.
\end{aligned}$$

To estimate the right hand side of (4.21), we show several elementary lemmas.

We will estimate the right hand side of (4.21) in $L_x^2(|x| < t)$. Thanks to (4.4) and Lemma 4.2, we have

$$(4.22) \quad \left\| \lambda c_1 s^{-\frac{5}{2}} \langle \mu \rangle^{\frac{5}{2}} (\tilde{\rho}(s, \mu)^{\frac{2}{3}} - \rho(\mu)^{\frac{2}{3}}) \rho(\mu) e^{i(\alpha+\beta)} \right\|_{L_x^2(|x| < t)} \lesssim t^{-1-\frac{5}{6}}.$$

Furthermore, it follows from Lemma 4.3 that

$$(4.23) \quad \|\operatorname{Re}(J_5 + J_7)\|_{L_x^2(|x| < t)} \lesssim t^{-1-\frac{5}{6}} \log t.$$

By (4.4) and (4.17), we obtain

$$(4.24) \quad \|\operatorname{Re} I_1\|_{L^2(|x| < t)} \lesssim t^{-2} \|\langle \cdot \rangle^{-1} \rho\|_{L_\mu^2(\mathbb{R}^3)} \leq t^{-2} \|\rho\|_{L_\mu^2}.$$

Similarly,

$$\begin{aligned}
(4.25) \quad \|\operatorname{Re} J_6\|_{L^2(|x| < t)} & \lesssim t^{-2} (\|\langle \cdot \rangle^{-\frac{3}{5}} \rho\|_{L_\mu^{\frac{10}{3}}(\mathbb{R}^3)}^{\frac{5}{3}} + \|\langle \cdot \rangle^{-2} \rho\|_{L_\mu^2(\mathbb{R}^3)}) \\
& \leq t^{-2} \|\rho\|_{H^1} \langle \|\rho\|_{H^1} \rangle^{\frac{2}{3}}.
\end{aligned}$$

To estimate J_1 , we note that

$$\begin{aligned}
& \partial_s^2 e^{i\tilde{\Psi}(s, \mu) \log s} \\
& = i \partial_s^2 \tilde{\Psi}(s, \mu) \log s e^{i\tilde{\Psi}(s, \mu) \log s} + 2i s^{-1} \partial_s \tilde{\Psi}(s, \mu) e^{i\tilde{\Psi}(s, \mu) \log s} \\
& \quad - (\partial_s \tilde{\Psi}(s, \mu) \log s)^2 e^{i\tilde{\Psi}(s, \mu) \log s} - 2s^{-1} \tilde{\Psi}(s, \mu) \partial_s \tilde{\Psi}(s, \mu) \log s e^{i\tilde{\Psi}(s, \mu) \log s} \\
& \quad - i s^{-2} \tilde{\Psi}(s, \mu) e^{i\tilde{\Psi}(s, \mu) \log s} - s^{-2} \tilde{\Psi}(s, \mu)^2 e^{i\tilde{\Psi}(s, \mu) \log s}.
\end{aligned}$$

Then, one sees from Lemma 4.3 that

$$\begin{aligned}
\|J_1\|_{L^2(|x| < t)} & \lesssim t^{-2-\frac{5}{6}} \log t + t^{-2-\frac{5}{6}} + t^{-2-\frac{7}{6}} (\log t)^2 \\
& \quad + t^{-3} (\log t) \|\langle \cdot \rangle^{-4} \rho^{\frac{1}{3}}\|_{L_\mu^2(\mathbb{R}^3)} + t^{-2} \|\langle \cdot \rangle^{-1} \tilde{\rho}(\cdot, t)^{\frac{2}{3}} \rho\|_{L_\mu^2(\mathbb{R}^3)} \\
& \quad + t^{-2} \|\langle \cdot \rangle^{-1} \tilde{\rho}(\cdot, t)^{\frac{4}{3}} \rho\|_{L_\mu^2(\mathbb{R}^3)}.
\end{aligned}$$

Using (4.5) and the Hölder and the Sobolev inequalities, we obtain

$$(4.26) \quad \|J_1\|_{L^2(|x| < t)} \lesssim t^{-2} \langle \|\rho\|_{H^1} \rangle^{\frac{7}{3}}.$$

We next estimate J_3 and J_4 . To this end, we remark that

$$\begin{aligned}
& \partial_j \partial_k (\langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta} e^{i\tilde{\Psi}(s, \mu) \log s}) \\
& = (\partial_j \partial_k (\langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta})) e^{i\tilde{\Psi}(s, \mu) \log s} \\
& \quad + i (\partial_j (\langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta})) \partial_k \tilde{\Psi}(s, \mu) (\log s) e^{i\tilde{\Psi}(s, \mu) \log s}
\end{aligned}$$

$$\begin{aligned}
& + i(\partial_k(\langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta})) \partial_j \tilde{\Psi}(s, \mu) (\log s) e^{i\tilde{\Psi}(s, \mu) \log s} \\
& + i\langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta} \partial_j \partial_k \tilde{\Psi}(s, \mu) (\log s) e^{i\tilde{\Psi}(s, \mu) \log s} \\
& - \langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta} \partial_j \tilde{\Psi}(s, \mu) \partial_k \tilde{\Psi}(s, \mu) (\log s)^2 e^{i\tilde{\Psi}(s, \mu) \log s}.
\end{aligned}$$

Hence, it follows from Lemma 4.4 that

$$\begin{aligned}
& \|J_3 + J_4\|_{L_x^2(|x|<t)} \\
& \lesssim t^{-2} \left(\left\| \langle \cdot \rangle^{\frac{3}{2}} \nabla^2 (\langle \cdot \rangle^{\frac{3}{2}} \rho e^{i\beta}) \right\|_{L_\mu^2} + (\log t) \left\| \langle \cdot \rangle^3 \rho^{\frac{2}{3}} |\nabla^2 \rho| \right\|_{L_\mu^2} \right) \\
& + t^{-1-\frac{5}{6}} (\log t) \left(\left\| \langle \cdot \rangle^{\frac{3}{2}} |\nabla (\langle \cdot \rangle^{\frac{3}{2}} \rho e^{i\beta})| |\langle \cdot \rangle^{\frac{1}{2}} \nabla \rho| \right\|_{L_\mu^2} + \left\| \langle \mu \rangle^{\frac{7}{2}} |\nabla \rho|^2 \right\|_{L_\mu^2} \right) \\
& + t^{-\frac{7}{3}} (\log t) \left(\left\| \langle \cdot \rangle^{-\frac{1}{2}} |\nabla (\langle \cdot \rangle^{\frac{3}{2}} \rho e^{i\beta})| \right\|_{L_\mu^2} + \|\rho\|_{L^2} \right) \\
& + t^{-2} (\log t)^2 \left\| \langle \cdot \rangle^3 |\nabla \rho|^2 \right\|_{L_\mu^2} + t^{-3} (\log t)^2 \left\| \langle \cdot \rangle^{-2} \rho^{\frac{1}{3}} \right\|_{L_\mu^2}.
\end{aligned}$$

Thus,
(4.27)

$$\|\operatorname{Re}(J_3 + J_4)\|_{L_x^2(|x|<t)} \lesssim t^{-1-\frac{5}{6}} (\log t) \left\langle \left\| \langle \cdot \rangle^3 \rho e^{i\beta} \right\|_{H^2} + \left\| \langle \cdot \rangle^3 \rho \right\|_{H^2} \right\rangle^2.$$

Finally, we estimate J_2 . Since

$$\begin{aligned}
& \partial_{\mu_j} \partial_s (\langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta} e^{i\tilde{\Psi}(s, \mu) \log s}) \\
& = (\partial_{\mu_j} (\langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta})) (i \partial_s \tilde{\Psi}(s, \mu) \log s + i s^{-1} \tilde{\Psi}(s, \mu)) e^{i\tilde{\Psi}(s, \mu) \log s} \\
& + \langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta} (i \partial_{\mu_j} \partial_s \tilde{\Psi}(s, \mu) \log s + i s^{-1} \partial_{\mu_j} \tilde{\Psi}(s, \mu)) e^{i\tilde{\Psi}(s, \mu) \log s} \\
& - \langle \mu \rangle^{\frac{3}{2}} \rho(\mu) e^{i\beta} (\partial_s \tilde{\Psi}(s, \mu) \log s + s^{-1} \tilde{\Psi}(s, \mu)) \partial_{\mu_j} \tilde{\Psi}(s, \mu) (\log s) e^{i\tilde{\Psi}(s, \mu) \log s},
\end{aligned}$$

we deduce from Lemmas 4.3 and 4.4 that

$$\begin{aligned}
& \|J_2\|_{L_x^2(|x|<t)} \\
& \lesssim t^{-1-\frac{5}{6}} \left\| \langle \cdot \rangle^{5/2} \rho |\nabla \rho| \right\|_{L_\mu^2} + t^{-2} (\log t) \left\| \langle \cdot \rangle^2 \rho^{\frac{4}{3}} |\nabla \rho| \right\|_{L_\mu^2} \\
& + t^{-2} \left\| \langle \cdot \rangle^{1/2} |\nabla (\langle \cdot \rangle^{3/2} \rho e^{i\beta})| \rho^{\frac{2}{3}} \right\|_{L_\mu^2} + t^{-\frac{13}{6}} (\log t)^2 \left\| \langle \cdot \rangle^{\frac{3}{2}} \rho |\nabla \rho| \right\|_{L_\mu^2} \\
& + t^{-\frac{7}{3}} (\log t) (\|\rho\|_{L_\mu^2} + \|\langle \cdot \rangle |\nabla \rho| \|_{L_\mu^2} + \|\langle \cdot \rangle^{-1/2} \nabla (\langle \cdot \rangle^{3/2} \rho e^{i\beta}) \|_{L_\mu^2}) \\
& + t^{-\frac{7}{3}} \left(\|\rho\|_{L_\mu^2} + \left\| \langle \cdot \rangle^{-1/2} |\nabla (\langle \cdot \rangle^{3/2} \rho e^{i\beta})| \right\|_{L_\mu^2} \right) + t^{-\frac{8}{3}} (\log t)^2 \left\| \langle \cdot \rangle^{-1} \rho \right\|_{L_\mu^2},
\end{aligned}$$

from which we obtain

$$(4.28) \quad \|J_2\|_{L_x^2(|x|<t)} \lesssim t^{-1-\frac{5}{6}} \left\langle \left\| \langle \cdot \rangle^3 \rho e^{i\beta} \right\|_{H^2} + \left\| \langle \cdot \rangle^3 \rho \right\|_{H^2} \right\rangle^2.$$

Substituting (4.22), (4.23), (4.24), (4.25), (4.26), (4.27), (4.28) into (4.21), we obtain (4.15) because $\|\langle \cdot \rangle^3 \rho e^{i\beta}\|_{H^2} + \|\langle \cdot \rangle^3 \rho\|_{H^2} \lesssim \|(\phi_0, \phi_1)\|_Y$. This completes the proof of Proposition 4.5. \square

Next we give an estimate for difference between $(\square + 1)\tilde{v}_{\text{ap}}$ and the non-resonance part N_{nr} .

Proposition 4.6. *For all $n \geq 2$ and $t \geq 3$, we have*

$$(4.29) \quad \|(\square + 1)v_n - N_n\|_{L_x^2(|x| < t)} \lesssim |c_n| t^{-2} \langle \|(\phi_0, \phi_1)\|_Y \rangle^{\frac{7}{3}}.$$

In particular,

$$\|(\square + 1)\tilde{v}_{\text{ap}} - N_{\text{nr}}\|_{L_x^2(|x| < t)} \lesssim t^{-2} \langle \|(\phi_0, \phi_1)\|_Y \rangle^{\frac{7}{3}}.$$

Proof. Denoting $d_n = -\lambda c_n/(n^2 - 1)$, we have

$$(\square + 1)v_n = d_n \operatorname{Re}(\square + 1)(s^{-\frac{5}{2}} \langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\langle \mu \rangle^{-1}s + in\tilde{\Psi}(\mu, t) \log s}).$$

As in the previous case, we split

$$(4.30) \quad \begin{aligned} & d_n(\square + 1)(s^{-\frac{5}{2}} \langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\langle \mu \rangle^{-1}s + in\tilde{\Psi}(s, \mu) \log s}) \\ &= d_n((\square + 1)s^{-\frac{5}{2}} e^{in\langle \mu \rangle^{-1}s}) \langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s, \mu) \log s} \\ &+ d_n s^{-\frac{5}{2}} e^{in\langle \mu \rangle^{-1}s} \square(\langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s, \mu) \log s}) \\ &+ 2d_n \partial_t(s^{-\frac{5}{2}} e^{in\langle \mu \rangle^{-1}s}) \partial_t(\langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s, \mu) \log s}) \\ &- 2d_n \nabla_x(s^{-\frac{5}{2}} e^{in\langle \mu \rangle^{-1}s}) \cdot \nabla_x(\langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s, \mu) \log s}) \\ &=: I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n}. \end{aligned}$$

By means of Lemma 4.1,

$$(4.31) \quad \begin{aligned} I_{1,n} &= \lambda c_n s^{-\frac{5}{2}} \langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in(\alpha+\beta)} \\ &- 2ind_n s^{-\frac{7}{2}} \langle \mu \rangle^{\frac{7}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in(\alpha+\beta)} \\ &+ \frac{35d_n}{4} s^{-\frac{9}{2}} \langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in(\alpha+\beta)} \\ &=: \lambda c_n s^{-\frac{5}{2}} \langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in(\alpha+\beta)} + K_{1,n} + K_{2,n}. \end{aligned}$$

In a similar way, we have

$$(4.32) \quad \begin{aligned} I_{2,n} &= d_n s^{-\frac{5}{2}} e^{in\langle \mu \rangle^{-1}s} \langle \mu \rangle^{\frac{5}{2}} \rho(\mu) e^{in\beta} \partial_s^2(\tilde{\rho}(s, \mu)^{\frac{2}{3}} e^{in\tilde{\Psi}(s, \mu) \log s}) \\ &- 2d_n s^{-\frac{7}{2}} e^{in\langle \mu \rangle^{-1}s} \langle \mu \rangle^2 \mu \cdot \nabla_\mu \partial_s(\langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s, \mu) \log s}) \\ &- d_n s^{-\frac{9}{2}} e^{in\langle \mu \rangle^{-1}s} \langle \mu \rangle^2 \Delta_\mu(\langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s, \mu) \log s}) \\ &- 3d_n s^{-\frac{9}{2}} e^{in\langle \mu \rangle^{-1}s} \langle \mu \rangle^2 \mu \cdot \nabla_\mu(\langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s, \mu) \log s}) \\ &- d_n s^{-\frac{9}{2}} e^{in\langle \mu \rangle^{-1}s} \langle \mu \rangle^2 \sum_{1 \leq i, j \leq 3} \mu_i \mu_j \partial_i \partial_j(\langle \mu \rangle^{\frac{5}{2}} \tilde{\rho}(s, \mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s, \mu) \log s}) \\ &=: J_{1,n} + J_{2,n} + J_{3,n} - \frac{3}{2} K_{3,n} + J_{4,n}. \end{aligned}$$

By using the identities

$$\begin{aligned} \partial_t(s^{-\frac{5}{2}} e^{in\langle \mu \rangle^{-1}s}) &= in\langle \mu \rangle s^{-\frac{5}{2}} e^{in\langle \mu \rangle^{-1}s} - \frac{5}{2} s^{-\frac{7}{2}} e^{in\langle \mu \rangle^{-1}s}, \\ \nabla_x e^{in\langle \mu \rangle^{-1}s} &= -in\mu e^{in\langle \mu \rangle^{-1}s}, \end{aligned}$$

we obtain

$$\begin{aligned}
(4.33) \quad I_{4,n} &= 2ind_n s^{-\frac{5}{2}} e^{in\langle\mu\rangle^{-1}s} \\
&\quad \times \mu \cdot \{s^{-1}\langle\mu\rangle\nabla_\mu + s^{-1}\langle\mu\rangle\mu(\mu \cdot \nabla_\mu)\} \\
&\quad (\langle\mu\rangle^{\frac{5}{2}}\tilde{\rho}(s,\mu)^{\frac{2}{3}}\rho(\mu)e^{in\beta}e^{in\tilde{\Psi}(s,\mu)\log s}) \\
&= 2ind_n s^{-\frac{7}{2}} e^{in\langle\mu\rangle^{-1}s}\langle\mu\rangle^3 \\
&\quad \times (\mu \cdot \nabla_\mu (\langle\mu\rangle^{\frac{5}{2}}\tilde{\rho}(s,\mu)^{\frac{2}{3}}\rho(\mu)e^{in\beta}e^{in\tilde{\Psi}(s,\mu)\log s}))
\end{aligned}$$

and

$$\begin{aligned}
(4.34) \quad I_{3,n} &= -2n^2 d_n s^{-\frac{7}{2}} \langle\mu\rangle e^{in\langle\mu\rangle^{-1}s} \tilde{\Psi}(s,\mu) \langle\mu\rangle^{\frac{5}{2}} \tilde{\rho}(s,\mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s,\mu)\log s} \\
&\quad - 2n^2 d_n s^{-\frac{5}{2}} (\log s) \langle\mu\rangle e^{in\langle\mu\rangle^{-1}s} \\
&\quad \times \partial_s \tilde{\Psi}(s,\mu) \langle\mu\rangle^{\frac{5}{2}} \tilde{\rho}(s,\mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s,\mu)\log s} \\
&\quad + 2ind_n s^{-\frac{5}{2}} \langle\mu\rangle e^{in\langle\mu\rangle^{-1}s} \langle\mu\rangle^{\frac{5}{2}} (\partial_s \tilde{\rho}(s,\mu)^{\frac{2}{3}}) \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s,\mu)\log s} \\
&\quad - 2ind_n s^{-\frac{7}{2}} e^{in\langle\mu\rangle^{-1}s} \\
&\quad \times \langle\mu\rangle^3 (\mu \cdot \nabla_\mu (\langle\mu\rangle^{\frac{5}{2}} \tilde{\rho}(s,\mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s,\mu)\log s})) \\
&\quad - 5ind_n s^{-\frac{9}{2}} e^{in\langle\mu\rangle^{-1}s} \tilde{\Psi}(s,\mu) \langle\mu\rangle^{\frac{5}{2}} \tilde{\rho}(s,\mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s,\mu)\log s} \\
&\quad - 5ind_n s^{-\frac{7}{2}} (\log s) e^{in\langle\mu\rangle^{-1}s} \\
&\quad \times \partial_s \tilde{\Psi}(s,\mu) \langle\mu\rangle^{\frac{5}{2}} \tilde{\rho}(s,\mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s,\mu)\log s} \\
&\quad - 5d_n s^{-\frac{7}{2}} e^{in\langle\mu\rangle^{-1}s} \langle\mu\rangle^{\frac{5}{2}} (\partial_s \tilde{\rho}(s,\mu)^{\frac{2}{3}}) \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s,\mu)\log s} \\
&\quad + 5d_n s^{-\frac{9}{2}} e^{in\langle\mu\rangle^{-1}s} \langle\mu\rangle^2 (\mu \cdot \nabla_\mu (\langle\mu\rangle^{\frac{5}{2}} \tilde{\rho}(s,\mu)^{\frac{2}{3}} \rho(\mu) e^{in\beta} e^{in\tilde{\Psi}(s,\mu)\log s})) \\
&=: K_{4,n} + J_{5,n} + K_{5,n} - I_{4,n} + J_{6,n} + J_{7,n} + K_{6,n} + \frac{5}{2} K_{3,n}.
\end{aligned}$$

Substituting (4.31), (4.32), (4.33) and (4.34) into (4.30), we conclude that

(4.35)

$$\begin{aligned}
(\square + 1)v_n - N_n &= \lambda c_n s^{-\frac{5}{2}} \langle\mu\rangle^{\frac{5}{2}} \left(\tilde{\rho}(s,\mu)^{\frac{2}{3}} - \rho(\mu)^{\frac{2}{3}} \right) \rho(\mu) \operatorname{Re}(e^{in(\alpha+\beta)}) \\
&\quad + \sum_{k=1}^7 \operatorname{Re} J_{k,n} + \sum_{k=1}^6 \operatorname{Re} K_{k,n}.
\end{aligned}$$

The first term of the right hand side is $O(|c_n|t^{-1-5/6})$ in $L_x^2(|x| < t)$ with the help of (4.4) and Lemma 4.2. The estimates for $J_{k,n}$ are similar to those for the corresponding J_k . The difference are that the additional decay effect of order $O(t^{-1})$ make them all higher order terms, that each term is multiplied by $\langle\mu\rangle\tilde{\rho}(s,\mu)^{2/3}$, and that the order in n is at most $O(n^2|d_n|) = O(|c_n|) = O(n^{-8/3})$ as $n \rightarrow \infty$ because the phase parts are differentiated at most twice. The terms $K_{k,n}$ are new but the estimates for $K_{k,n}$ are done in a similar way. This completes the proof of Proposition 4.6. \square

Acknowledgments. S.M. is partially supported by the Sumitomo Foundation, Basic Science Research Projects No. 161145. J.S. is partially supported by JSPS, Grant-in-Aid for Young Scientists (A) 25707004.

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